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Energy and momentum integrals for progressive capillary–gravity waves

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Definitions of energy density, energy flux and momentum flux for capillary–gravity waves are derived by integration of the equations of motion and also by Whitham's averaged Lagrangian method. We then confirm recent results due to Hogan (1979) both in the general case and in the case of pure capillary waves. Comparison with the Lagrangian results also allows us to give general definitions of 'wave action density' and 'wave action flux'.

1. Introduction

In a recent paper Hogan (1979) gives some general results for energy and momentum integrals for capillary–gravity waves, and corresponding specific calculations for pure capillary waves. In checking the results for pure capillary waves the present author derived them from Whitham's averaged Lagrangian method, a fairly simple technique when the Lagrangian is already known, as it is in this case. The definitions of the various quantities in terms of the Lagrangian are derived here in § 3, but before that the mass, momentum and energy equations are derived in § 2 by integration through the depth of the fluid, bringing in the surface tension through the surface pressure, and then distributing the surface tension terms into the energy density and energy and momentum fluxes. We thus produce integral definitions and averaged Lagrangian definitions for the same quantities.

Comparison of the definitions with Hogan's general results is made in § 4, and one additional result is given for the S_{22} component of the radiation stress tensor (see (13) for the definition). We also give physical interpretations of the various definitions. By looking at the Lagrangian definitions we can define the 'wave action density' and 'wave action flux' in terms of integral properties. These quantities have been shown to have considerable importance in slowly varying wave-trains.

Finally in § 5, we show how to obtain Hogan's results for pure capillary waves by the Lagrangian method.

2. Mass, momentum and energy equations

We define axes $(\mathbf{x}, y)$, $\mathbf{x} = (x_1, x_2)$ with y vertically upward; the fluid is bounded below by $y = -h(\mathbf{x})$ and the free surface is at $y = \eta(\mathbf{x}, t)$, with mean value $y = b(\mathbf{x}, t)$. There is a mainstream flow $(\mathbf{U}(\mathbf{x}, t), 0)$ on which is superimposed a wave motion with velocity field $(\mathbf{u}(\mathbf{x}, y, t), v(\mathbf{x}, y, t))$; vector quantities have horizontal components only. The functions $h(\mathbf{x})$, $\mathbf{U}(\mathbf{x}, t)$, $b(\mathbf{x}, t)$ and the mean properties of the wave motion are assumed to be slowly varying compared to the variations of $\mathbf{u}(\mathbf{x},y,t)$, $v(\mathbf{x},y,t)$ and $\eta(\mathbf{x},t)$.

The definition of the wave motion is made specific by assuming that the local mean horizontal velocity in the wave, $\bar{\mathbf{u}}$, is zero. If we further assume that the wave motion is irrotational this will be true at all depths. The lack of any vertical component in the mainstream flow imposes limitations on the type of flow which is admissible, restricting it to a flow which satisfies 'shallow water equations' in which the vertical acceleration is negligible. Extension to flows with significant vertical accelerations has been considered to some extent by Peregrine (1976, § II F) and a rather different approach by Hasselmann (1971) also covers this possibility. However, a complete development of this case is beyond the scope of the present paper.

We follow generally the method of Phillips (1966, § 3), which is to integrate the equations of continuity, momentum and energy over the vertical from $y = -h$ to $y = \eta$, to take $\partial/\partial t$ and $\partial/\partial x_i$ derivatives outside the integrals using the appropriate theorem, and to use the kinematic surface and bed conditions

$$\left.\begin{aligned} \frac{\partial \eta}{\partial t} + (U_i + u_i)\frac{\partial \eta}{\partial x_i} &= v \quad \text{on} \quad y = \eta, \\ -(U_i + u_i)\frac{\partial h}{\partial x_i} &= v \quad \text{on} \quad y = -h, \end{aligned}\right\} \tag{1}$$

to cancel many of the extra terms which appear. Here the summation convention is applied. There are differences both of notation and definition from Phillips, which will appear as we go along.

Starting with the continuity equation

$$\frac{\partial U_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i} + \frac{\partial v}{\partial y} = 0, \tag{2}$$

the integration and derivative exchange gives

$$\begin{aligned} \frac{\partial}{\partial x_i}\int_{-h}^{\eta}(U_i + u_i)\,dy - (U_i + u_i)_\eta \frac{\partial \eta}{\partial x_i} \\ -(U_i + u_i)_{-h}\frac{\partial h}{\partial x_i} + v_\eta - v_{-h} = 0, \end{aligned} \tag{3}$$

where a suffix η or $-h$ implies evaluation at the surface or bed. Using (1) and (2), and noting that U_i is independent of y, this becomes

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x_i}\big(U_i(\eta + h)\big) + \frac{\partial}{\partial x_i}\int_{-h}^{\eta} u_i\,dy = 0. \tag{4}$$

Multiplying by density ρ and averaging over the waves we then have the mass continuity equation

$$\rho\frac{\partial b}{\partial t} + \frac{\partial}{\partial x_i}\big((\rho U_i d) + I_i\big) = 0 \tag{5}$$

where

$$d(x_i, t) = b + h \tag{6}$$

is the mean total depth, and

$$I_i = \overline{\rho\int_{-h}^{\eta} u_i\,dy} \tag{7}$$

is the wave momentum in the i-direction, equal to the mass flux in that direction due to the waves. Here the overbar denotes the average over the waves.

The horizontal Euler equations are written in the form

$$\frac{\partial}{\partial t}\rho\,(U_i+u_i)+\frac{\partial}{\partial x_j}\left(\rho\,(U_i+u_i)\,(U_j+u_j)+p\delta_{ij}\right)+\frac{\partial}{\partial y}\left(\rho\,(U_i+u_i)\,v\right)=0, \tag{8}$$

and the same procedure applied. Using (1) and (2) we find on averaging that

$$\frac{\partial}{\partial t}(\rho U_i d)+\frac{\partial I_i}{\partial t}+\frac{\partial}{\partial x_j}(\rho U_i U_j d)+\frac{\partial}{\partial x_j}(U_j I_i+U_i I_j) + \frac{\partial}{\partial x_j}\overline{\int_{-h}^{\eta}(\rho u_i u_j+p\delta_{ij})\,dy}-\overline{p_\eta\frac{\partial\eta}{\partial x_i}}-\overline{p_{-h}}\frac{\partial h}{\partial x_i}=0. \tag{9}$$

We define

$$S^g_{ij}=\overline{\int_{-h}^{\eta}(\rho u_i u_j+p\delta_{ij})\,dy}-\tfrac{1}{2}\rho g d^2\delta_{ij} \tag{10}$$

as the radiation stress tensor for pure gravity waves. This represents the excess momentum flux due to the waves, in the absence of surface tension. The definition differs from that of Phillips (3.6.12) but agrees (for S_{11}) with that of Longuet-Higgins (1975, (1.6)). Subtracting U_i times equation (5) from (9) we obtain

$$\frac{\partial I_i}{\partial t}+\frac{\partial}{\partial x_j}(U_j I_i+S^g_{ij})+\rho d\left(\frac{\partial U_i}{\partial t}+U_j\frac{\partial U_i}{\partial x_j}+g\frac{\partial d}{\partial x_i}\right) + I_j\frac{\partial U_i}{\partial x_j}+\tau\overline{J\frac{\partial\eta}{\partial x_i}}-\overline{p_{-h}}\frac{\partial h}{\partial x_i}=0 \tag{11}$$

as the averaged momentum equation of the wave motion. Here τ is the surface tension coefficient and J is the sum of the principal curvatures of the surface:

$$J=\left\{\left(1+\left(\frac{\partial\eta}{\partial x_1}\right)^2\right)\frac{\partial^2\eta}{\partial x_2^2}-2\frac{\partial\eta}{\partial x_1}\frac{\partial\eta}{\partial x_2}\frac{\partial^2\eta}{\partial x_1\partial x_2}+\left(1+\left(\frac{\partial\eta}{\partial x_2}\right)^2\right)\frac{\partial^2\eta}{\partial x_1^2}\right\} \times\left(1+\left(\frac{\partial\eta}{\partial x_1}\right)^2+\left(\frac{\partial\eta}{\partial x_2}\right)^2\right)^{-\frac{3}{2}}. \tag{12}$$

This surface tension term properly belongs in with S^g_{ij} as a momentum flux due to the waves. The appropriate expression for S_{ij} giving both $\partial S^g_{ij}/x_i$ and $\overline{\tau J\partial\eta/\partial x_i}$ is

$$\begin{aligned}S_{ij}&=S^g_{ij}+S^\tau_{ij}\\ &=S^g_{ij}+\tau\left(\overline{\frac{\dfrac{\partial\eta}{\partial x_i}\dfrac{\partial\eta}{\partial x_j}}{\left(1+\dfrac{\partial\eta}{\partial x_k}\dfrac{\partial\eta}{\partial x_k}\right)^{\frac{1}{2}}}}-\overline{\left(\left(1+\frac{\partial\eta}{\partial x_k}\frac{\partial\eta}{\partial x_k}\right)^{\frac{1}{2}}-1\right)}\delta_{ij}\right).\end{aligned} \tag{13}$$

Differentiation with respect to x_j will show that this satisfies the equation, and it is a tensor since the square root in each term is invariant under rotation of axes. It was originally obtained in another way, which we shall see below.

To find the mean bottom pressure we follow Phillips and write the vertical momentum equation as

$$\frac{\partial}{\partial t}(\rho v)+\frac{\partial}{\partial x_j}(\rho(U_j+u_j)\,v)+\frac{\partial}{\partial y}(\rho v^2)+\frac{\partial p}{\partial y}+\rho g=0 \tag{14}$$

which, applying the usual treatment, becomes

$$\rho \frac{\partial}{\partial t}\overline{\int_{-h}^{\eta} v\,dy}+\rho\frac{\partial}{\partial x_j}\left(U_j\overline{\int_{-h}^{\eta} v\,dy}\right)+\rho\frac{\partial}{\partial x_j}\overline{\int_{-h}^{\eta} u_j v\,dy}-\tau\overline{J}-\overline{p_{-h}}+\rho g(b+h) = 0. \quad (15)$$

For periodic progressive waves the mean values of the integrals and of J vanish. This is most easily seen by considering a two-dimensional wave. Then η is symmetrical about the crest, so is u (and therefore I_i does not vanish), but v is anti-symmetrical, so the integrals over half a wavelength on each side of the crest cancel. The average value of J can be expressed as a function of $\partial\eta/\partial x_1 = \eta'$ vanishing because η' is the same at each limit (cf. Hogan (1979), equation after (2.17)). Because of the slowly varying assumption the waves are locally plane, so the argument applies in general. However, we must note that if there is any standing wave component these integrals do not vanish.

Thus $\overline{p_{-h}} = \rho g d$, i.e. the mean bottom pressure is hydrostatic, and with (13) equation (11) becomes

$$\frac{\partial I_i}{\partial t}+\frac{\partial}{\partial x_j}(U_j I_i+S_{ij})+\rho d\left(\frac{\partial U_i}{\partial t}+U_j\frac{\partial U_i}{\partial x_j}+g\frac{\partial b}{\partial x_i}\right)+I_j\frac{\partial U_i}{\partial x_j} = 0. \quad (16)$$

The energy equation is obtained by taking the scalar product of the momentum equations and the velocity, and can be written as

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho(U_j+u_j)(U_j+u_j)+\tfrac{1}{2}\rho v^2+\rho g y\right)\\&\quad+\frac{\partial}{\partial x_i}\left\{(U_i+u_i)\left(\tfrac{1}{2}\rho(U_j+u_j)(U_j+u_j)+\tfrac{1}{2}\rho v^2+\rho g y+p\right)\right\}\\&\quad+\frac{\partial}{\partial y}\left\{v\left(\tfrac{1}{2}\rho(U_j+u_j)(U_j+u_j)+\tfrac{1}{2}\rho v^2+\rho g y+p\right)\right\}\\&\quad= 0. \end{aligned} \quad (17)$$

Integration and the kinematic boundary conditions give

$$\begin{aligned}&\frac{\partial}{\partial t}\int_{-h}^{\eta}\left(\tfrac{1}{2}\rho(U_j+u_j)(U_j+u_j)+\tfrac{1}{2}\rho v^2+\rho g y\right)dy\\&\quad+\frac{\partial}{\partial x_i}\int_{-h}^{\eta}(U_i+u_i)\left(\tfrac{1}{2}\rho(U_j+u_j)(U_j+u_j)+\tfrac{1}{2}\rho v^2+\rho g y+p\right)dy\\&\quad-\tau J\frac{\partial\eta}{\partial t} = 0.\end{aligned} \quad (18)$$

We define the energy density for gravity waves as

$$E^g = \overline{\int_{-h}^{\eta}\tfrac{1}{2}\rho(u_j u_j+v^2)\,dy}+\tfrac{1}{2}\rho g(\overline{\eta^2}-b^2) \quad (19)$$

$$= \overline{\int_{-h}^{\eta}\left(\tfrac{1}{2}\rho(u_j u_j+v^2)+\rho g y\right)dy}-\tfrac{1}{2}\rho g(b^2-h^2), \quad (20)$$

and the energy flux vector for gravity waves as

$$F_i^g = \overline{\int_{-h}^{\eta} u_i\left(\tfrac{1}{2}\rho(u_j u_j + v^2) + p + \rho g(y-b)\right) dy} \tag{21}$$

$$= \overline{\int_{-h}^{\eta} u_i\left(\tfrac{1}{2}\rho(u_j u_j + v^2) + p + \rho g y\right) dy} - gbI_i. \tag{22}$$

Averaging (18) and subtracting from it $(\frac{1}{2}U_i U_i + gb)$ times (5) and U_i times (11) gives

$$\frac{\partial E^g}{\partial t} + \frac{\partial}{\partial x_i}(U_i E^g + F_i^g) + S_{ij}^g \frac{\partial U_j}{\partial x_i} + I_i\left(\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + g\frac{\partial b}{\partial x_i}\right)$$
$$- \overline{\tau J\left(\frac{\partial \eta}{\partial t} + U_i \frac{\partial \eta}{\partial x_i}\right)} + \overline{p_{-h}}\, U_i \frac{\partial h}{\partial x_i} - U_i \rho g d \frac{\partial h}{\partial x_i} = 0. \tag{23}$$

The hydrostatic mean pressure on the bed allows the last two terms to cancel. The surface tension term has to be re-arranged into appropriate parts E^τ, F_i^τ and S_{ij}^τ which are given by

$$E = E^g + E^\tau = E^g + \tau \overline{\left(\left(1 + \frac{\partial \eta}{\partial x_k}\frac{\partial \eta}{\partial x_k}\right)^{\frac{1}{2}} - 1\right)}, \tag{24}$$

$$F_i = F_i^g + F_i^\tau = F_i^g - \tau \overline{\left(\frac{\partial \eta}{\partial x_i}\right)\left(\frac{\partial \eta}{\partial t} + U_j \frac{\partial \eta}{\partial x_j}\right)\left(1 + \frac{\partial \eta}{\partial x_k}\frac{\partial \eta}{\partial x_k}\right)^{-\frac{1}{2}}} \tag{25}$$

$$= F_i^g - \tau \overline{\frac{\partial \eta}{\partial x_i}\left(v - u_j \frac{\partial \eta}{\partial x_j}\right)_\eta \left(1 + \frac{\partial \eta}{\partial x_k}\frac{\partial \eta}{\partial x_k}\right)^{-\frac{1}{2}}} \tag{26}$$

using (1), with S_{ij}^τ given by (13).

The averaged energy equation for the wave motion is finally

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_i}(U_i E + F_i) + S_{ij}\frac{\partial U_j}{\partial x_i} + I_i\left(\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + g\frac{\partial b}{\partial x_i}\right) = 0. \tag{27}$$

In the next section we show how to derive equations equivalent to (5), (16) and (27) from Whitham's averaged Lagrangian method; then in § 4 we go on to discuss the surface tension terms E^τ, F_i^τ and S_{ij}^τ and compare them with earlier work.

3. Whitham's Lagrangian method

To find an averaged Lagrangian for Whitham's method we start from the Lagrangian formulation of the water waves equations due to Luke (1967), and add on a surface tension term. The Euler equations for

$$L = -\rho \int_{-h}^{\eta} (\Phi_t + \tfrac{1}{2}(\Phi_{x_i}\Phi_{x_i} + \Phi_y^2) + gy)\, dy - \tau((1 + \eta_{x_i}\eta_{x_i})^{\frac{1}{2}} - 1) \tag{28}$$

(where suffices t, x_i denote derivatives) give both Laplace's equation and the boundary conditions. The only difference from Luke's paper is in the surface pressure boundary condition which follows immediately from the η Euler equation

$$\frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial \eta_{x_i}}\right) - \frac{\partial L}{\partial \eta} = 0. \tag{29}$$

Now consider a uniform wave-train with waves propagating in the x_1-direction so that we can write

$$\left.\begin{aligned} \Phi &= U_i x_i - \gamma t + \phi(kx_1 - \omega t), \\ \eta &= b + \eta(kx_1 - \omega t), \end{aligned}\right\} \tag{30}$$

where U_i, γ, k and b are constants. Substituting into L and averaging over a wave-length we have

$$\begin{aligned} \mathscr{L} = \rho(\gamma - \tfrac{1}{2}U_i U_i)\, d - \rho \overline{\int_{-h}^{\eta}\left(\frac{\partial \phi}{\partial t} + U_1 \frac{\partial \phi}{\partial x_1}\right) dy} \\ - E^g - \tfrac{1}{2}\rho g(b^2 - h^2) - \tau \overline{\left(\left(1 + \left(\frac{\partial \eta}{\partial x_1}\right)^2\right)^{\frac{1}{2}} - 1\right)}. \end{aligned} \tag{31}$$

Now

$$\frac{\partial \phi}{\partial t} = -\frac{\omega}{k}\frac{\partial \phi}{\partial x_1} \tag{32}$$

$$= -(c + U_1)\frac{\partial \phi}{\partial x_1} \tag{33}$$

where $c = \sigma/k$ is the phase speed of the waves relative to the flow. The integral term in (31) is thus

$$+\rho c \overline{\int_{-h}^{\eta} \frac{\partial \phi}{\partial x_1} dy} = cI_1 = 2T \tag{34}$$

where T is the kinetic energy, by Longuet-Higgins (1975, (B)). The E^g and τ terms combine to $T + V$, where V is the potential energy, and (31) becomes

$$\mathscr{L} = \rho(\gamma - \tfrac{1}{2}U_i U_i)\, d - \tfrac{1}{2}\rho g(b^2 - h^2) + T - V, \tag{35}$$

which may be compared with Whitham (1967, (23)). The extra term $\frac{1}{2}\rho g h^2$ can in fact be dropped as it does not contribute to any of Whitham's equations. Written in this form (35) will hold if the waves now propagate in some other direction. We shall write

$$\mathscr{L}^w = T - V, \tag{36}$$

because it is simply a property of the wave motion

$$\mathscr{L}^w = \mathscr{L}^w(\sigma, k, a, d) \tag{37}$$

$$= \mathscr{L}^w(\omega - U_i k_i, k, a, d), \tag{38}$$

where a is a measure of the wave amplitude, and $k^2 = k_i k_i$. Note that $\sigma/2\pi$ is the frequency which waves of this length would have in the absence of any free stream U_i, whilst $\omega/2\pi$ is the actual frequency in our axes. For differentiations in Whitham's method we have to use the form (38).

Whitham gives the following eight equations derived by variational methods from $\mathscr{L}$:

$$\text{(i)}\ \frac{\partial \mathscr{L}}{\partial a} = 0, \quad \text{i.e.}\ \frac{\partial \mathscr{L}^w}{\partial a} = 0, \tag{39}$$

the dispersion relation;

$$\text{(ii)}\ \frac{\partial \mathscr{L}}{\partial b} = 0, \quad \text{i.e.}\ \rho(\gamma - \tfrac{1}{2}U_i U_i) - \rho g b + \frac{\partial \mathscr{L}^w}{\partial d} = 0; \tag{40}$$

$$\text{(iii)}\ \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial \gamma}\right) - \frac{\partial}{\partial x_i}\left(\frac{\partial \mathscr{L}}{\partial U_i}\right) = 0,$$

$$\text{i.e.}\quad \frac{\partial}{\partial t}(\rho b) + \frac{\partial}{\partial x_i}\left(\rho d U_i + k_i \frac{\partial \mathscr{L}^w}{\partial \sigma}\right) = 0, \tag{41}$$

the integrated mass continuity equation, equivalent to (5);

$$\text{(iv)}\ \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial \omega}\right) - \frac{\partial}{\partial x_i}\left(\frac{\partial \mathscr{L}}{\partial k_i}\right) = 0,$$

$$\text{i.e.}\quad \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}^w}{\partial \sigma}\right) + \frac{\partial}{\partial x_i}\left(U_i \frac{\partial \mathscr{L}^w}{\partial \sigma} - \frac{k_i}{k}\frac{\partial \mathscr{L}^w}{\partial k}\right) = 0; \tag{42}$$

and four consistency equations

$$\frac{\partial \omega}{\partial x_i} + \frac{\partial k_i}{\partial t} = 0, \quad \frac{\partial k_1}{\partial x_2} = \frac{\partial k_2}{\partial x_1}, \quad \frac{\partial \gamma}{\partial x_i} + \frac{\partial U_i}{\partial t} = 0, \quad \frac{\partial U_1}{\partial x_2} = \frac{\partial U_2}{\partial x_1}. \tag{43}$$

By comparing (5) and (41) we can immediately define

$$I_i = k_i \frac{\partial \mathscr{L}^w}{\partial \sigma}. \tag{44}$$

None of these equations are momentum or energy equations as they stand, although the full set are equivalent to them. To derive the momentum equation we consider

$$\frac{\partial \mathscr{L}^w}{\partial x_i} = \frac{\partial \mathscr{L}^w}{\partial \sigma}\frac{\partial \sigma}{\partial x_i} + \frac{\partial \mathscr{L}^w}{\partial k}\frac{\partial k}{\partial x_i} + \frac{\partial \mathscr{L}^w}{\partial d}\frac{\partial d}{\partial x_i} \tag{45}$$

(noting (39)). Then using $\sigma = \omega - U_j k_j$, (42) and (43), we can re-arrange (45) as

$$\frac{\partial}{\partial t}\left(k_i \frac{\partial \mathscr{L}^w}{\partial \sigma}\right) + \frac{\partial}{\partial x_j}\left(U_j k_i \frac{\partial \mathscr{L}^w}{\partial \sigma} - \frac{k_i k_j}{k}\frac{\partial \mathscr{L}^w}{\partial k} + \mathscr{L}^w \delta_{ij}\right) + k_j \frac{\partial \mathscr{L}^w}{\partial \sigma}\frac{\partial U_i}{\partial x_j} - \frac{\partial \mathscr{L}^w}{\partial d}\frac{\partial d}{\partial x_i} = 0. \tag{46}$$

If we write

$$S_{ij} = -\left(\frac{k_i k_j}{k}\frac{\partial \mathscr{L}^w}{\partial k} + d\frac{\partial \mathscr{L}^w}{\partial d}\delta_{ij} - \mathscr{L}^w \delta_{ij}\right) \tag{47}$$

equation (46) is

$$\frac{\partial I_i}{\partial t} + \frac{\partial}{\partial x_j}(U_j I_i + S_{ij}) + d\frac{\partial}{\partial x_i}\frac{\partial \mathscr{L}^w}{\partial d} + I_j \frac{\partial U_i}{\partial x_j} = 0. \tag{48}$$

Deriving $\partial \mathscr{L}^w / \partial d$ from (40) and replacing $\partial \gamma / \partial x_i$ from (43) makes (48) exactly the same as the earlier momentum equation (16).

For the energy equation we similarly consider

$$\frac{\partial \mathscr{L}^w}{\partial t} = \frac{\partial \mathscr{L}^w}{\partial \sigma}\frac{\partial \sigma}{\partial t} + \frac{\partial \mathscr{L}^w}{\partial k}\frac{\partial k}{\partial t} + \frac{\partial \mathscr{L}^w}{\partial d}\frac{\partial b}{\partial t}. \tag{49}$$

This time rather more algebra is required and we eventually arrive at

$$\frac{\partial}{\partial t}\left(\sigma\frac{\partial \mathscr{L}^w}{\partial \sigma} - \mathscr{L}^w\right) + \frac{\partial}{\partial x_i}\left(U_i\left(\sigma\frac{\partial \mathscr{L}^w}{\partial \sigma} - \mathscr{L}^w\right) - \sigma\frac{k_i}{k}\frac{\partial \mathscr{L}^w}{\partial k}\right) + S_{ij}\frac{\partial U_i}{\partial x_j} - \frac{1}{\rho}\frac{\partial \mathscr{L}^w}{\partial d}\frac{\partial I_i}{\partial x_i} = 0, \tag{50}$$

with S_{ij} and I_i given by (47) and (44). If we write

$$E = \sigma\frac{\partial \mathscr{L}^w}{\partial \sigma} - \mathscr{L}^w \tag{51}$$

and

$$F_i = -\frac{\sigma k_i}{k}\frac{\partial \mathscr{L}^w}{\partial k} - \frac{1}{\rho}\frac{\partial \mathscr{L}^w}{\partial d}I_i, \tag{52}$$

the energy equation becomes

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_i}(U_i E + F_i) + S_{ij}\frac{\partial U_i}{\partial x_j} + \frac{I_i}{\rho}\frac{\partial}{\partial x_i}\left(\frac{\partial \mathscr{L}^w}{\partial d}\right) = 0, \tag{53}$$

exactly equivalent to (27).

Note that Whitham's method assumes that the main stream flow U_i is irrotational, whereas the method of § 2 does not. Also in this section we have that since in the case of infinite depth $\mathscr{L}^w$ is independent of d, (40) becomes

$$\rho(\gamma - \tfrac{1}{2}U_j U_j) - \rho g b = 0, \tag{54}$$

or, when differentiated with respect to x_i and using (43),

$$\frac{\partial U_i}{\partial t} + U_j\frac{\partial U_j}{\partial x_j} + g\frac{\partial b}{\partial x_i} = 0; \tag{55}$$

i.e. for infinite depth, variations in the main stream are driven only by variations in the slope of the mean free surface, or *vice versa*. This is equivalent to the assumption of hydrostatic pressure in the main stream.

4. The surface tension terms

In a recent paper Hogan (1979) defines E, F_1 and S_{11} for capillary–gravity waves and evaluates various formulae, in particular in the special case of pure capillary waves in water of infinite depth for which he uses the present author's exact solution (Crapper 1957).

We now show that his definitions are correct, and give some physical interpretation.

The formulae (13), (24) and (26) have been found in a purely mathematical way, and in the general two-dimensional formulation S_{ij} in particular is extremely complicated. The present author originally derived them from the un-averaged Lagrangian L (28) using formulae from Whitham ((1965), (9), (10)). There, in the absence of any free stream U_i, E, F_1 and S_{11} (using the symbols temporarily for the un-averaged forms) are given as

$$\left.\begin{aligned} E &= \Phi_t \frac{\partial L}{\partial \Phi_t} + \eta_t \frac{\partial L}{\partial \eta_t} - L, \\ F_i &= \Phi_t \frac{\partial L}{\partial \Phi_{x_i}} + \eta_t \frac{\partial L}{\partial \eta_{x_i}}, \\ S_{ij} &= -\left(\Phi_{x_j} \frac{\partial L}{\partial \Phi_{x_i}} + \eta_{x_j} \frac{\partial L}{\partial \eta_{x_i}} - L\delta_{ij}\right). \end{aligned}\right\} \tag{56}$$

The surface tension term in L is a function of η_{x_i} only, so only the last terms in E and F_i and the last two in S_{ij} contribute, giving the required forms.

To make comparison with Hogan we now assume that the waves are propagating in the x direction, with $\partial/\partial x_2 \equiv 0$. The surface tension contribution to the energy is

$$E^\tau = V^\tau = \tau \overline{\left(\left(1 + \left(\frac{\partial \eta}{\partial x_1}\right)^2\right)^{\frac{1}{2}} - 1\right)}, \tag{57}$$

which is the usual formula of τ times the relative extension of a surface element. The contribution to the flux is

$$F_1^\tau = \overline{\frac{-\tau \dfrac{\partial \eta}{\partial x_1}\left(v - u \dfrac{\partial \eta}{\partial x_1}\right)_\eta}{\left(1 + \left(\dfrac{\partial \eta}{\partial x_1}\right)^2\right)^{\frac{1}{2}}}}, \quad F_2^\tau = 0, \tag{58}$$

in agreement with Hogan. If we write this as

$$F_1^\tau = \tau \overline{\left(\left(1 + \left(\frac{\partial \eta}{\partial x_1}\right)^2\right)^{\frac{1}{2}} - 1\right) u_\eta} + \overline{\tau(1 - \cos\zeta)\, u_\eta} - \overline{\tau \sin \zeta v_\eta} \tag{59}$$

where $\tan \zeta = \partial \eta/\partial x_1$, we can interpret the terms as a transport of energy E^τ at speed u_η plus the rate of working of the surface force τ pulling on the fluid to the right of the point x_1 in a direction making an angle ζ below the negative x_1 axis. This may be compared with the other part of the energy flux definition (21) which shows a transport of kinetic energy, a transport of gravitational potential energy and the work done by the pressure acting on the fluid on the right. It seems therefore that the physical interpretation is correct.

The surface tension part of the radiation stress (13) is now

$$S_{11}^\tau = \tau \overline{\left[\frac{\left(\dfrac{\partial \eta}{\partial x_1}\right)^2}{\left(1 + \left(\dfrac{\partial \eta}{\partial x_1}\right)^2\right)^{\frac{1}{2}}} - \left(1 + \left(\frac{\partial \eta}{\partial x_1}\right)^2\right)^{\frac{1}{2}} + 1\right]} \tag{60}$$

$$= \tau \overline{\left[1 - \frac{1}{\left(1 + \left(\dfrac{\partial \eta}{\partial x_1}\right)^2\right)^{\frac{1}{2}}}\right]} \tag{61}$$

$$= \tau(1 - \overline{\cos \zeta}) \tag{62}$$

agreeing with Hogan;

$$S_{12}^\tau = S_{21}^\tau = 0; \tag{63}$$

and

$$S_{22}^\tau = -E^\tau = -V^\tau. \tag{64}$$

The S_{11} component is just the force $-\tau\cos\zeta$, normalized to be zero for no waves and averaged, comparable with the pressure force p in S^g_{11}, so again is as expected. The S_{22} term agrees to second order with the definition in Longuet-Higgins & Stewart (1964, (17)). It arises because the 2-direction force on a line drawn in the surface parallel to x_1 is increased by an amount equal to the increase in the length of the line due to the waves.

Hogan finds the following formulae from these definitions

$$\left.\begin{aligned} F_1 &= (3T-2V^g)\,c+B(I_1/\rho+ch), \\ S_{11} &= 4T-3V^g-V^\tau+2Bh, \end{aligned}\right\} \tag{65}$$

in our notation, where $B=\tfrac{1}{2}\rho\overline{u^2_{-h}}$ is a Bernoulli constant, the waves are progressing in the x_1 direction with velocity c, and $U_i=0$, $b=0$. We here show that

$$S_{22}=T-V^g-V^\tau+Bh \tag{66}$$

to complete the set. Considering (10) we have

$$S_{11}=\overline{\int_{-h}^{\eta}(\rho u_1^2+p)\,dy}-\tfrac{1}{2}\rho gh^2+S^\tau_{11}, \tag{67}$$

$$S_{22}=\overline{\int_{-h}^{\eta}p\,dy}-\tfrac{1}{2}\rho gh^2+S^\tau_{22}, \tag{68}$$

and from Hogan's equation (2.19)

$$F_1=\frac{BI_1}{\rho}+\rho c\overline{\int_{-h}^{\eta}u_1^2\,dy}+F^\tau_1. \tag{69}$$

Thus

$$S_{22}=S^\tau_{22}+S_{11}-\overline{\int_{-h}^{\eta}\rho u_1^2\,dy}-S^\tau_{11}. \tag{70}$$

Now from (25)

$$F^\tau_1=\frac{-\tau\overline{\dfrac{\partial\eta}{\partial t}\dfrac{\partial\eta}{\partial x_1}}}{\left(1+\left(\dfrac{\partial\eta}{\partial x_1}\right)^2\right)^{\frac{1}{2}}}=\overline{\frac{\tau c\left(\dfrac{\partial\eta}{\partial x_1}\right)^2}{\left(1+\left(\dfrac{\partial\eta}{\partial x_1}\right)^2\right)^{\frac{1}{2}}}} \tag{71}$$

$$=\tau c\overline{\left[\left(1+\left(\frac{\partial\eta}{\partial x_1}\right)^2\right)^{\frac{1}{2}}-\frac{1}{\left(1+\left(\dfrac{\partial\eta}{\partial x_1}\right)^2\right)^{\frac{1}{2}}}\right]} \tag{72}$$

$$=c(V^\tau+S^\tau_{11}). \tag{73}$$

So using (69), (64), (65) and (73) we have the result (66). Considering appropriate rotations of axes we then have the general form

$$S_{ij}=(3T-2V^g+Bh)\frac{k_ik_j}{k^2}+(T-V^g-V^\tau+Bh)\,\delta_{ij}; \tag{74}$$

the infinite depth gravity wave limit of this result was given by Peregrine & Thomas (1979).

The Lagrangian formulation of § 2 cannot be checked directly against these results unless a particular Lagrangian is specified. Nevertheless it can easily be seen that for consistency we require

$$\left.\begin{aligned} \frac{\partial \mathscr{L}^w}{\partial \sigma} &= \frac{2T}{\sigma}; \\ -\frac{\partial \mathscr{L}^w}{\partial k} &= \frac{1}{k}\{(3T-2V^g)+Bh\}; \\ \frac{\partial \mathscr{L}^w}{\partial d} &= -B. \end{aligned}\right\} \quad (75)$$

For these x_1-direction waves equation (40) shows that

$$\frac{\partial \mathscr{L}^w}{\partial d} = -\rho(\gamma - \tfrac{1}{2}c^2) \quad (76)$$

which in fact agrees with Hogan's definition of B, with $\rho\gamma$ as the Bernoulli constant of a steady flow on which these waves are stationary. The results (75) are also consistent with a differential relation for $d\mathscr{L}^w$ proved by Longuet-Higgins (1975, (4.17)). Although his proof omits surface tension, it is easy to see that if it is included the terms actually cancel very early in the proof, thus providing a further check on our results.

The quantity $\partial \mathscr{L}^w/\partial \sigma$ is usually known as the 'wave action density' and (for x_1-direction waves) $-\partial \mathscr{L}^w/\partial k$ is the 'wave action flux'. Through Whitham's equation (42) these quantities are seen to have direct importance in slowly varying situations, and (75) gives physical meaning to them in a general nonlinear context. For small waves these definitions agree with those of Whitham (1974, (16.82), (16.83)). For deep-water gravity waves they have in fact been given previously by Peregrine & Thomas (1979). It is interesting to note that they depend only indirectly on the surface tension.

5. Pure capillary waves in infinite-depth fluid

To compare the results derived from Whitham's theory with those of Hogan we use the averaged Lagrangian

$$\mathscr{L}^w = 2\tau - \frac{\rho\sigma^2}{k^3} - \frac{\tau^2 k^3}{\rho\sigma^2} \quad (77)$$

originally derived by Lighthill (1965) using the results of Crapper (1957). The differentiations are straightforward, and then substitution of the formula

$$\frac{k}{\rho c^2} = \frac{k^3}{\rho\sigma^2} = \frac{1}{\tau}\left(\frac{1+A^2}{1-A^2}\right), \quad (78)$$

also derived from Crapper (1957), is seen to lead directly to Hogan's results for I_1, E, F_1 and S_{11}. Additionally

$$S_{22} = \mathscr{L}^w = -\frac{\tau 4A^4}{1-A^4}. \quad (79)$$

In checking Hogan's results the present author originally derived all of them by this method. It is generally speaking much simpler than the integrals in the general definition of S_{11} and F_1, although of course the energy integrals have to be calculated to give $\mathscr{L}^w$. With the work of Cokelet (1977) it should be possible to make a good approximate Lagrangian for gravity waves, and the formulae for the various integral properties will make checks on its accuracy, through the derivatives with respect to σ, k and d.

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